# Existence and uniqueness of solutions for second order discrete systems of two-point BVP


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**Abstract**

We study existence and uniqueness of solutions for second-order discrete systems of two-point boundary value problem. The approach is based on Perov′s fixed point theorem and Schauder′ fixed point theorem in generalized Banach spaces. To illustrate our theory, some examples are presented.



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## 1 Introduction

The purpose of this paper is to study the following discrete boundary value problems (BVP for short). Let $f, g : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous,

$$\begin{aligned} \frac{\Delta^2 x_i}{\tau^2} &= f\left(t_i, x_i, y_i\right),\ i = 1, 2 \ldots, n-1, \\ \frac{\Delta^2 y_i}{\tau^2} &= g\left(t_i, x_i, y_i\right),\ i = 1, 2 \ldots, n-1, \end{aligned} \tag{1.1}$$

$$\begin{aligned} u\frac{\Delta x_0}{\tau} + v\frac{\Delta x_n}{\tau} &= 0,\ u + v \neq 0, \\ \bar{u}\frac{\Delta y_0}{\tau} + \bar{v}\frac{\Delta y_n}{\tau} &= 0,\ \overline{u} + \overline{v} \neq 0, \end{aligned} \tag{1.2}$$

where $\tau = \frac{N}{n} < N$; the grid points are denoted by $t_i = i\tau$ for $\Delta^2 x_i = x_{i+1} - 2x_i + x_{i-1}$ for $i = 1, 2, \ldots, n.; u, v, \overline{u}, \overline{v}$ are constants.

BVP serves as a mathematical model and arise in the study of solid state physics, combinatorial analysis, chemical reactions, population dynamics, and so forth. Recently, the existence of solutions for BVPs of difference equations, with various boundary conditions has received increasing attention from many authors (see for example, [4, 5, 6, 7, 8, 9, 10, 11, 14, 16]). Coskun and Demir [4] investigated the existence and uniqueness of solutions of the BVPs

$$\begin{aligned} \frac{\Delta x_i}{\tau} &= f\left(t_i, x_i\right) \qquad i = 0, 1 \ldots, n-1, \\ u x_0 + v x_n &= w, \qquad u + v \neq 0, \end{aligned} \tag{1.3}$$


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and

$$\frac{\Delta^2 x_i}{\tau^2} = f(t_i, x_i) \qquad i = 0, 1 \dots, n-1,$$
$$u x_0 + v x_n = 0, \qquad u + v \neq 0, \tag{1.4}$$
$$\bar{u}\frac{\Delta x_0}{\tau} + \bar{v}\frac{\Delta x_n}{\tau} = 0, \qquad \overline{u} + \overline{v} \neq 0.$$

Banach contraction principle was used in [4] to prove te existence and uniqueness results. In literature, Bolojan et al. [3] studied the existence of solutions to initial value problems for non linear first order differential systems with non linear non local boundary conditions of functional type. Perov, Schauder, and Leray-Schauder were used to prove the existence results.

However, to the best of our knowledge, little work has been done on difference equation in generalized Banach spaces. By following the method used in Bolojan et al. [3], we develop new results of the BVP (1.1) - (1.2) by applying Perov′s fixed point theorem and Schauder′ fixed point theorem in generalized Banach spaces. Thus we extend and generalize some of the results of Coskun and Demir [4].

This paper is organized as follows. In section 2, we introduce some notations, definitions and basic results. In section 3, we state and prove our main results by using Perov′s and Schauder′ fixed point theorem.

## 2 Preliminary results

In this section, we will give some basic lemmas, definitions and theorem that will be used throughout this paper.

**Definition 2.1.** By a vector-valued metric on X we mean a mapping $d : X \times X \to \mathbb{R}^n$ such that

(i) $d(u, v) \geq 0$ for all $u, v \in X$ and if $d(u, v) = 0$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbb{R}^n, x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$ by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \dots, n$. We call the pair $(X, d)$ a *generalized metric space* with

$$d(\mathbf{x}, \mathbf{y}) := \begin{pmatrix} d_1(\mathbf{x}, \mathbf{y}) \\ . \\ . \\ . \\ d_n(\mathbf{x}, \mathbf{y}) \end{pmatrix},$$

where $d_i, i = 1, 2, \dots, n$ is a metric on $X$. Notice that $d$ is generalized metric space on $X$ if and only if $d_i, i = 1, 2, \dots, n$ are metrics on $X$.

Let $X := \mathbb{R}^{n+1}$. We consider the vector-valued norm

$$\| (\mathbf{x}, \mathbf{y}) \| = \begin{bmatrix} | \mathbf{x} | \\ | \mathbf{y} | \end{bmatrix}, \tag{2.1}$$

for $(\mathbf{x},\mathbf{y}) \in X \times X$. Also $\mid \mathbf{x} \mid = \max_{i=0,\dots,n} \mid x_i \mid$ for $\mathbf{x} \in X$, and define

$$d(\mathbf{x},\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\| \qquad \text{for all } \mathbf{x},\mathbf{y} \in X.$$

The pair $(X,d)$ is called a generalized Banach space. For such a space convergence and completeness are similar to those in usual metric spaces.

**Definition 2.2.** A square matrix $M$ with the non negative elements is said to be convergent to zero if

$$M^k \to 0 \text{ as } k \to \infty.$$

The property of being convergent to zero is equivalent to the following conditions from the characterisation lemma below (see [1, 2, 12, 13, 15, 17]).

**Lemma 2.3.** Let M be a square matrix of non negative numbers. The following statements are equivalent:

(i) $M$ is a matrix convergent to zero;

(ii) $I-M$ is non singular and $(I-M)^{-1} = I+M+M^2+\dots$ (where $I$ stands for the unit matrix of the same order as $M$);

(iii) the eigenvalues of $M$ are located inside the unit disc of the complex plane;

(iv) $I-M$ is non singular and $(I-M)^{-1}$ has non negative elements.

Note that, according to the equivalence of the statement $(i)$ and $(iv)$, a matrix $M$ is convergent to zero if and only if the matrix $I-M$ is inverse-positive.

**Definition 2.4.** Let $(X,d)$ be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix $M$ such that

$$d(N(x),N(y)) \leq Md(x,y), \forall x,y \in X. \tag{2.2}$$

**Theorem 2.5.** (Schauder). Let $X$ be a Banach space, $D \subset X$ a non empty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e. $T$ is continuous and $T(D)$ is relatively compact. Then $T$ has at least one fixed point).

## 3 Existence results

In this section, we present the existence of solutions to the problem (1.1) - (1.2) follows from Perov's fixed point theorem in case that non linearity $f,g$ and the functional $a_i,b_i, i=1,2$ satisfy Lipschitz condition.
The problem (1.1) - (1.2) can be rewrite as a system of summation equation of the form

$$\begin{aligned} x_i &= h^2 \sum_{j=1}^{n-1}\sum_{k=1}^{n} G_1(i,j,k)\, f(t_j,x_j,y_j), \qquad i=0,1\dots,n, \\ y_i &= h^2 \sum_{j=1}^{n-1}\sum_{k=1}^{n} G_2(i,j,k)\, f(t_j,x_j,y_j), \qquad i=0,1\dots,n, \end{aligned} \tag{3.1}$$

which is explicitly given by [4],

$$G_1\left(i,j,k\right)=\begin{cases}\dfrac{u}{u+v}G(k,j) & \text{for } 1\leq k\leq i,\\ -\dfrac{v}{u+v}G(k,j) & \text{for } i+1\leq k\leq n,\end{cases}$$

and

$$G_2\left(i,j,k\right)=\begin{cases}\dfrac{\overline{u}}{\overline{u}+\overline{v}}G(k,j) & \text{for } 1\leq k\leq i,\\ -\dfrac{\overline{v}}{\overline{u}+\overline{v}}G(k,j) & \text{for } i+1\leq k\leq n.\end{cases}$$

The system (3.1) can be viewed as the following fixed point problem

$$\mathbf{T}(\mathbf{x},\mathbf{y})=(\mathbf{x},\mathbf{y}),$$

for all $(\mathbf{x},\mathbf{y})\in X\times X$ so that

$$\mathbf{T}(\mathbf{x},\mathbf{y})=\begin{pmatrix}T_1(\mathbf{x},\mathbf{y})_0,\ldots,T_1(\mathbf{x},\mathbf{y})_n\\ T_2(\mathbf{x},\mathbf{y})_0,\ldots,T_2(\mathbf{x},\mathbf{y}_n\end{pmatrix}.$$

We define the operator $\mathbf{T}$ in a component wise, where

$$T_1(\mathbf{x},\mathbf{y})_i:=h^2\sum_{j=1}^{n-1}\sum_{k=1}^{n}G_1\left(i,j\right)f\left(t_j,x_j,y_j\right),\qquad i=0,1,\ldots,n,$$
$$T_2(\mathbf{x},\mathbf{y})_i=h^2\sum_{j=1}^{n-1}\sum_{k=1}^{n}G_2\left(i,j\right)g\left(t_j,x_j,y_j\right),\qquad i=0,1,\ldots,n,$$

for all $(\mathbf{x},\mathbf{y})\in X\times X$. Also define

$$S_l=\max\mid G_l\left(i,j\right)\mid$$

for $l=1,2$.

We state and prove our theorem concerning the existence and uniqueness of solution to the problems (1.1) - (1.2).

**Theorem 3.1.** Let $f,g:[0,N]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous and $u+v\neq 0,\bar{u}+\bar{v}\neq 0$. There are constants $a_1,a_2,b_1,b_2>0$ such that

$$|f\left(t,x,y\right)-f\left(t,\overline{x},\overline{y}\right)|\leq a_1|x-\overline{x}|+b_1|y-\overline{y}|,\tag{3.2}$$
$$|g\left(t,x,y\right)-g\left(t,\overline{x},\overline{y}\right)|\leq a_2|x-\overline{x}|+b_2|y-\overline{y}|,\tag{3.3}$$

for all $t\in[0,N],(x,y)\in\mathbb{R}^2$.

Assume that the matrix

$$M=\begin{bmatrix}mS_1a_1 & mS_1b_1\\ mS_2a_2 & mS_2b_2\end{bmatrix},\tag{3.4}$$

is convergent to zero. Then the problem (1.1) - (1.2) has a unique solution.

*Proof.* Define the operator

$$\mathbf{T} = (T_1, T_2) : X \times X \to X \times X,$$

where $T_1, T_2$ are given by

$$T_1(\mathbf{x}, \mathbf{y})_i := h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} G_1(i,j) f(t_j, x_j, y_j), \qquad i = 0, 1, \dots, n,$$
$$T_2(\mathbf{x}, \mathbf{y})_i = h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} G_2(i,j) g(t_j, x_j, y_j), \qquad i = 0, 1, \dots, n,$$

for all $(\mathbf{x}, \mathbf{y}) \in X \times X$.

We prove that $T$ is contractive with respect to the convergent to zero matrix $M$, precisely

$$\|\mathbf{T}(\mathbf{x}, \mathbf{y}) - \mathbf{T}(\overline{\mathbf{x}}, \overline{\mathbf{y}})\| \le \begin{pmatrix} h^2 n S_1 a_1 & h^2 n S_1 b_1 \\ h^2 n S_2 a_2 & h^2 n S_2 b_2 \end{pmatrix} \begin{bmatrix} | \mathbf{x} - \overline{\mathbf{x}} | \\ | \mathbf{y} - \overline{\mathbf{y}} | \end{bmatrix}.$$

We have

$$\begin{aligned} |T_1(\mathbf{x}, \mathbf{y})_i - T_1(\overline{\mathbf{x}}, \overline{\mathbf{y}})_i| &\le h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} |G_1(i,j)| \left| f(t_j, x_j, y_j) - f\left(t_j, \overline{x}_j, \overline{y}_j\right) \right| \\ &\le h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} |G_1(i,j)| \left[ a_1 \mid x_j - \overline{x}_j \mid + b_1 \mid y_j - \overline{y}_j \mid \right] \\ &\le h^2 n S_1 \left[ a_1 \mid \mathbf{x} - \overline{\mathbf{x}} \mid + b_1 \mid \mathbf{y} - \overline{\mathbf{y}} \mid \right] \end{aligned} \tag{3.5}$$

for $i = 0, 1, \dots, n$. Similarly we obtain

$$\begin{aligned} |T_2(\mathbf{x}, \mathbf{y})_i - T_2(\overline{\mathbf{x}}, \overline{\mathbf{y}})_i| &\le h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} |G_2(i,j)| \left| g(t_j, x_j, y_j) - g\left(t_j, \overline{x}_j, \overline{y}_j\right) \right| \\ &\le h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} |G_2(i,j)| \left[ a_2 \mid x_j - \overline{x}_j \mid + b_2 \mid y_j - \overline{y}_j \mid \right] \\ &\le h^2 n S_2 \left[ a_2 \mid \mathbf{x} - \overline{\mathbf{x}} \mid + b_2 \mid \mathbf{y} - \overline{\mathbf{y}} \mid \right], \end{aligned} \tag{3.6}$$

for $i = 0, 1, \dots, n$. We can rewrite both inequalities (3.5) and (3.6) equivalently as

$$\begin{bmatrix} |T_1(\mathbf{x}, \mathbf{y})_i - T_1(\overline{\mathbf{x}}, \overline{\mathbf{y}})_i| \\ |T_2(\mathbf{x}, \mathbf{y})_i - T_2(\overline{\mathbf{x}}, \overline{\mathbf{y}})_i| \end{bmatrix} \le \begin{pmatrix} h^2 n S_1 a_1 & h^2 n S_1 b_1 \\ h^2 n S_2 a_2 & h^2 n S_2 b_2 \end{pmatrix} \begin{bmatrix} | \mathbf{x} - \overline{\mathbf{x}} | \\ | \mathbf{y} - \overline{\mathbf{y}} | \end{bmatrix}$$

or using the vector-valued norm

$$\|\mathbf{T}(\mathbf{x}, \mathbf{y}) - \mathbf{T}(\overline{\mathbf{x}}, \overline{\mathbf{y}})\| \le M \begin{bmatrix} | \mathbf{x} - \overline{\mathbf{x}} | \\ | \mathbf{y} - \overline{\mathbf{y}} | \end{bmatrix}$$

with

$$M = \begin{pmatrix} h^2nS_1a_1 & h^2nS_1b_1 \\ h^2nS_2a_2 & h^2nS_2b_2 \end{pmatrix}.$$

The result follows now from Perov′s fixed point theorem. Q.E.D.

Next in this section, we show the existence of solution to the problem (1.1), (1.2) follows from Scahuder′s fixed point theorem in case $f, g$ satisfy a relaxed growth condition.

**Theorem 3.2.** Let $f, g : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. There are constants $a_1, a_2, b_1, b_2 > 0$ such that

$$|f(t, x, y)| \leq a_1|x| + b_1|y| + k_1, \tag{3.7}$$

$$|g(t, x, y)| \leq a_2|x| + b_2|y| + k_2, \tag{3.8}$$

for all $t \in [0, N]$, $(x, y) \in \mathbb{R}^2$. If the matrix $M$ is given in (3.4) is convergent to zero, then the problem (1.1) - (1.2) has at least one solution.

*Proof.* To apply the Schauder's fixed point theorem, we show that a non empty, bounded, closed and convex subset $B$ of $X \times X$, $\mathbf{T}(B) \subset B$. Applying (3.7) and (3.8), we get

$$\begin{aligned} |\, T_1(\mathbf{x}, \mathbf{y})_i \,| &= \left| h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} G_1(i, j) f(t_j, x_j, y_j) \right| \\ &\leq h^2nS_1 [a_1|x_j| + b_1|y_j| + k_1] \\ &\leq h^2nS_1 [a_1|\mathbf{x}| + b_1|\mathbf{y}|] + h^2nS_1k_1 \end{aligned}$$

for $i = 0, 1 \ldots, n$. Similarly we obtain

$$\begin{aligned} |\, T_2(\mathbf{x}, \mathbf{y})_i \,| &= \left| h^2 \sum_{j=1}^{n-1} \sum_{k=1}^{n} G_2(i, j) g(t_j, x_j, y_j) \right| \\ &\leq h^2nS_2 [a_2|x_j| + b_2|y_j| + k_2] \\ &\leq h^2nS_2 [a_2|\mathbf{x}| + b_2|\mathbf{y}|] + h^2nS_2k_2. \end{aligned}$$

$$\begin{bmatrix} |T_1(\mathbf{x}, \mathbf{y})_i| \\ |T_2(x, y)_i| \end{bmatrix} \leq M \begin{bmatrix} |\, \mathbf{x} \,| \\ |\, \mathbf{y} \,| \end{bmatrix} + \begin{bmatrix} c_0 \\ C_0 \end{bmatrix},$$

where $M$ is given by (3.4) and is assumed to be convergent to zero, $c_0 = h^2nS_1k_1$ and $C_0 = h^2nS_2k_2$. Next for $|\mathbf{x}| \leq R_1$ and $|\mathbf{y}| \leq R_2$, we show $|\, T_1(\mathbf{x}, \mathbf{y})_i \,| \leq R_1$, $|\, T_2(\mathbf{x}, \mathbf{y})_i \,| \leq R_2$ for $i = 0, \ldots, n$. To this end it is sufficient that

$$M \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} c_0 \\ C_0 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

whence

$$(1-M)^{-1}\begin{bmatrix} c_0 \\ C_0 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

Notice that $1-M$ is invertible and its inverse $(1-M)^{-1}$ has non negative element since $M$ is convergent to zero. If $B = B_1 \times B_2$, where

$$B_1 = \{\mathbf{x} \in X : |\mathbf{x}| \leq R_1\}$$

and

$$B_2 = \{\mathbf{y} \in X :| \mathbf{y}| \leq R_2\}$$

then $\mathbf{T}(B) \subset B$. Therefore $\mathbf{T}$ has a fixed point in B. Q.E.D.

## 4 Some examples

In what follows, we will give two examples that illustrate our theory.

**Example 4.1.** Consider the special case of (1.1), (1.2) with:

$$f(t,x,y) = \frac{1}{5}\cos^2 x + \frac{2}{3}\sin y + t,$$

$$g(t,x,y) = \frac{1}{4}\sin^2 x + \frac{1}{2}\cos y + t.$$

Consider the step size $\tau = \frac{1}{n}$ where $n = 10, u = 2, v = 4, \bar{u} = 3$ and $\bar{v} = 2$. We have $a_1 = \frac{1}{5}, b_1 = \frac{2}{3}, a_2 = \frac{1}{4}$ and $b_2 = \frac{1}{2}$. Hence

$$M = \begin{bmatrix} \frac{1}{75} & \frac{2}{45} \\ \frac{3}{200} & \frac{3}{100} \end{bmatrix} \tag{4.1}$$

Since the eigenvalues of $M$ are $\lambda_1 = 0.049, \lambda_2 = -0.005$, the matrix (4.1) is convergent to zero if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ and from Theorem 3.1 the associated discrete boundary value problem has at least one solution.

**Example 4.2.** Consider the special case of (1.1) - (1.2) with:

$$f(t,x,y) = \frac{1}{2}\cos^2 x + ay + t,$$

$$g(t,x,y) = a\sin^2 x + \frac{1}{2}y + t.$$

Consider the step size $\tau = \frac{1}{n}$ where $n = 10, u = 1, v = 2, \bar{u} = 4$ and $\bar{v} = 2$. We have $a_1 = \frac{1}{2}, b_1 = |a|, a_2 = |a|$ and $b_2 = \frac{1}{2}$. Hence

$$M = \begin{bmatrix} \frac{1}{30} & \frac{|a|}{15} \\ \frac{|a|}{15} & \frac{1}{30} \end{bmatrix} \tag{4.2}$$

Since the eigenvalues of $M$ are $\lambda_1 = \frac{|a|}{15} + \frac{1}{30}, \lambda_2 = -\frac{|a|}{15} + \frac{1}{30}$, the matrix (4.2) is convergent to zero if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. It is also known that a matrix of this type is convergent to zero if $\frac{|a|}{15} + \frac{1}{30} < 1$. This condition is explained in [13]. Therefore, if $|a| < \frac{29}{2}$, then the matrix (4.2) is convergent to zero and from Theorem 3.1 the associated discrete boundary value problem has a unique solution.

## References

[1] Berman, A., & Plemmons, R. J. (1994). *Nonnegative matrices in the mathematical sciences. Society for Industrial and Applied Mathematics.*

[2] Boucherif, A. (2001). *Differential equations with nonlocal boundary conditions.* Nonlinear Analysis: Theory, Methods & Applications, 47(4), 2419-2430.

[3] Bolojan, O., Infante, G., & Precup, R. (2015). *Existence results for systems with nonlinear coupled nonlocal initial conditions.* Mathematica Bohemica, 140(4), 371-384.

[4] Coskun, a. E., & Demir, a. L. I. (2020). *On the Existence and Uniqueness of Solutions for a First and Second-order Discrete Boundary Value Problem.* Journal of Mathematical Analysis, 11(2).

[5] Ding, Y., Xiu, J., and Wei, Z. (2014) *Positive solutions for a system of discrete boundary value problem.* Bulletin of the Malaysian Math. Sc. Soc., 1-15.

[6] Ding, Y., Xu, J., & Wei, Z. (2015). *Positive solutions for a system of discrete boundary value problem.* Bulletin of the Malaysian Mathematical Sciences Society, 38(3), 1207-1221.

[7] Du, Z. (2008). *Positive solutions for a second-order three-point discrete boundary value problem.* J. Appl. Math. Comput., 26, 219-231.

[8] Luca, R. (2018). *Positive solution for a semipositone nonlocal discrete boundary value problems.* J. Appl. Math. Lett. 92,54-61.

[9] Luca, R. (2019). *Existence of positive solution for a semipositone discrete boundary value problems.* J. Nonlin. Anal. Modell. and Contr. 24(4), 658-678.

[10] Mohamed, M., and Ismail, N. H. S. (2018) *Positive solutions of singular multi-point discrete boundary value problem*, AIP Conference Proceedings 1974(1):030001 DOI: 10.1063/1.5041645 Conference: PROCEEDING OF THE 25TH NATIONAL SYMPOSIUM ON MATHEMATICAL SCIENCES (SKSM25): Mathematical Sciences as the Core of Intellectual Excellence, June 2018.

[11] Mohamed, M., and Ismail, N. H. S. (2015) *Postive solutions for a singular second order discrete system with a parameter.* Far East Journal of mathematical Sciences (FJMS), Vol 96, No 7, pp. 913-931.

[12] Precup, R. (2013). *Methods in nonlinear integral equations.* Springer Science & Business Media.

[13] Precup, R. (2009). *The role of matrices that are convergent to zero in the study of semilinear operator systems.* Mathematical and Computer Modelling, 49(3-4),703-708.

[14] Sun, J. P., Zhao, Y. H., & Li, W. T. (2005). *Existence of positive solution for second-order nonlinear discrete system with parameter.* Mathematical and Computer Modelling, 41(4-5), 493-499.

[15] Varga, R. S. (1962). *Iterative analysis.* Prentice Hall, Englewood Cliffs, NJ.

[16] Azmi, W. W. A., & Mohamed, M. (2013, September). *Existence and multiplicity of positive solutions for singular second order Dirichlet boundary value problem.* In AIP Conference Proceedings (Vol. 1557, No. 1, pp. 66-71). American Institute of Physics.

[17] Webb, J. R., & Lan, K. Q. (2006). *Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type.* Topological Methods in Nonlinear Analysis, 27(1), 91-115.